Potentials in Electrodynamics

- In Electrostatics we got a lot of mileage out of working w/ potentials instead of solving for $\vec{E}$ & $\vec{B}$ directly.
- Can we still do this in Electrodynamics?
- Look @ Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\times\vec{E} + \partial\vec{B}/\partial t = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

- The eqn $\vec{\nabla}\cdot\vec{B}=0$ tells us we can still write $\vec{B}$ as the curl of a vector $\vec{A}$: $\vec{B} = \vec{\nabla}\times\vec{A}$
- But if $\partial\vec{B}/\partial t \neq 0$ then $\vec{\nabla}\times\vec{E}\neq 0$, so we can no longer write $\vec{E}$ as the gradient of a scalar pot.!
- However, since $\vec{B} = \vec{\nabla}\times\vec{A}$, we have

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\vec{E} + \vec{\nabla}\times\frac{\partial\vec{A}}{\partial t} = \vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$$

- Since the curl of $\vec{E}+\partial\vec{A}/\partial t$ is zero, we can write it as the gradient of a scalar potential.

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\Phi \Rightarrow \vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$

- So in Electrodynamics, the Maxwell eqns $\vec{\nabla}\cdot\vec{B}=0$ and $\vec{\nabla}\times\vec{E}+\partial\vec{B}/\partial t = 0$ tell us (via the Helmholtz theory of vector fields) that we can always find a scalar function Φ and a vector function $\vec{A}$ that are potentials for $\vec{E}$ & $\vec{B}$:

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\Phi(r,t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$$

In statics, where nothing depends on time, Φ is just V.

- Since this is electrodynamics, where ρ & $\vec{J}$ can change over time, we expect that Φ & $\vec{A}$ can be functions of both <u>position</u> ($\vec{r}$) and <u>time</u> (t).

- For electrostatics & magnetostatics we saw that the potentials are redundant. Both V & V + constant give the same $\vec{E}$. Likewise, $\vec{A}$ & $\vec{A} + \vec{\nabla} h$ (for any function h of position $\vec{r}$) give the same $\vec{B}$. Is this still true in electrodynamics?

- If we add $\vec{\nabla}\lambda(\vec{r},t)$ to $\vec{A}$, the magnetic field does not change...

$$\vec{\nabla} \times (\vec{A} + \vec{\nabla}\lambda) = \vec{\nabla} \times \vec{A} + \cancelto{0}{\vec{\nabla} \times (\vec{\nabla}\lambda)} = \vec{\nabla} \times \vec{A}$$

... but the electric field <u>does</u> change:

$$-\vec{\nabla}\Phi - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\lambda) = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\left(\frac{\partial \lambda}{\partial t}\right) \neq -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

- To keep $\vec{E}$ the same, changing $\vec{A}$ by $\vec{\nabla}\lambda(\vec{r},t)$ has to be accompanied by changing Φ by $-\frac{\partial \lambda(\vec{r},t)}{\partial t}$.

$$\Phi(\vec{r},t),\ \vec{A}(\vec{r},t) \longleftrightarrow \Phi(\vec{r},t) - \frac{\partial \lambda(\vec{r},t)}{\partial t},\ \vec{A}(\vec{r},t) + \vec{\nabla}\lambda(\vec{r},t)$$

- So there are lots of pairs of potentials that give the same $\vec{E}$ & $\vec{B}$. Whenever possible, we should ask if there are potentials Φ & $\vec{A}$ w/ some nice property that simplifies whatever problem we're working on.

- This is called "CHOOSING A GAUGE" or "FIXING THE GAUGE."

- We'll return to this in a moment!

- Now, two of our Maxwell eqns are equivalent to saying we can find potentials Φ & $\vec{A}$ for $\vec{E}$ & $\vec{B}$. What about the other two eqns?

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right) = -\nabla^2\Phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\varepsilon_0$$

$$\Rightarrow \quad \nabla^2\Phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0$$

$$\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{-\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})} - \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right) = \mu_0\vec{J}$$

$$\Rightarrow \quad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\mu_0\vec{J}$$

- A convenient choice of gauge that reveals an important property of these eqns is LORENZ GAUGE (not Lorentz!) where we only work w/ Φ & $\vec{A}$ that have the property

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \quad \Rightarrow \quad \begin{aligned} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -\rho/\varepsilon_0 \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\vec{J} \end{aligned}$$

- So in Lorenz gauge, these last two Maxwell eqns tell us that Φ & $\vec{A}$ satisfy WAVE EQUATIONS w/ ρ & $\vec{J}$ (respectively) as sources.

- Any change in ρ & $\vec{J}$ will cause a change in Φ & $\vec{A}$ that propagates @ speed $c = 1/\sqrt{\mu_0\varepsilon_0}$. If you wiggle a charge @ $x = 0$ then the potential @ $x = 1\,\text{m}$ learns about it $1\text{m}-0/c = 3.34\times10^{-8}$ s later!

- Note that these aren't "auxiliary" wave eqns like the ones satisfied by $\vec{E}$ & $\vec{B}$, where we can find sol'ns that don't satisfy Maxwell's Eqns. These <u>are</u> Maxwells eqns.

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \qquad \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\varepsilon_0$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

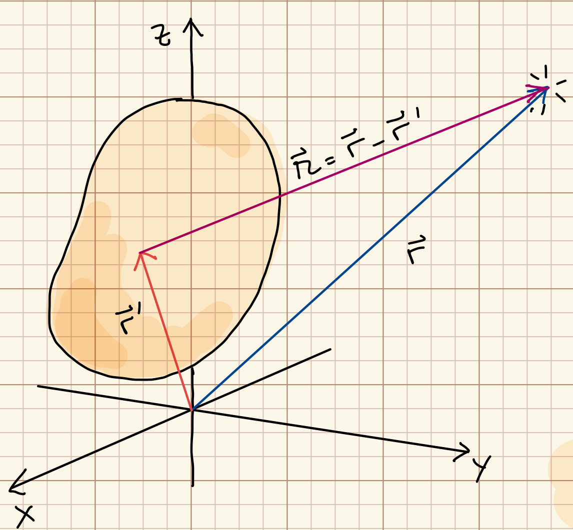
MAXWELL

$$\vec{F} = q\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right) + q\,\vec{v}\times(\vec{\nabla}\times\vec{A})$$

LORENTZ FORCE LAW

- Now here's something really remarkable. We can solve these equations as before (Coulomb integrals) w/ one small modification.

- The bit of charge dq @ $\vec{r}\,'$ contributes $(4\pi\varepsilon_0)^{-1}\,dq'/|\vec{r}-\vec{r}\,'|$ to Φ. But to find Φ @ position $\vec{r}$ & time t we need to know about dq' @ an <u>earlier time</u>.



← At time t this point learns what ρ & $\vec{J}$ were doing @ $\vec{r}\,'$ a short time <u>earlier</u>

$$t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$$

"RETARDED TIME"

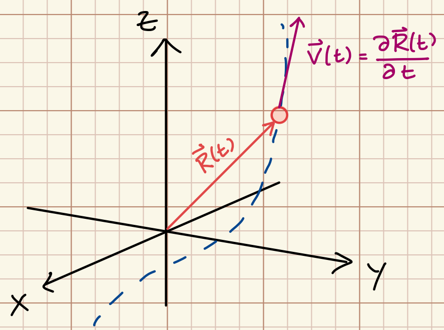
$$\Rightarrow \quad \Phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d\tau'\,\frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d\tau'\,\frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}$$

Visit every point $\vec{r}\,'$ w/ charge or current @ time $t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$ & add its contribution to Φ & $\vec{A}$.

- This is a marvelous result, but evaluating the integrals can be <u>complicated</u>.

- Let's look @ a "simple" example: A moving pt charge q w/ position $\vec{R}(t)$ @ time t.



$$\rho(\vec{r}, t) = q\, \delta^3(\vec{r} - \vec{R}(t))$$
$$\vec{J}(\vec{r}, t) = q\, \vec{v}(t)\, \delta^3(\vec{r} - \vec{R}(t))$$

- We can put these in our integrals for Φ & $\vec{A}$, and evaluate them. However, $\rho(\vec{r}\,', t_r) = q\, \delta^3(\vec{r}\,' - \vec{R}(t_r))$ and t_r is a function of $\vec{r}\,'$, so we have to be careful applying our Dirac delta rules!

$$\Phi(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \int d\tau' \frac{1}{\mathscr{r}}\, \delta^3(\vec{r}\,' - \vec{R}(t_r))$$

Change variable: $\vec{u} = \vec{r}\,' - \vec{R}(t_r)$ $\qquad \vec{v}(t_r) = \frac{\partial \vec{R}(t_r)}{\partial t_r}$

$$\hookrightarrow d\vec{u} = d\vec{r}\,' - \frac{\partial \vec{R}(t_r)}{\partial t_r}\, dt_r$$

$$t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c} \rightarrow dt_r = \vec{\nabla}'(t_r) \cdot d\vec{r}\,' = \vec{\nabla}'\left(-\frac{|\vec{r} - \vec{r}\,'|}{c}\right) \cdot d\vec{r}\,'$$
$$= \frac{1}{c\mathscr{r}}\, \vec{\mathscr{r}} \cdot d\vec{r}\,'$$

Uses the <u>Jacobian</u>

$$\hookrightarrow d\vec{u} = d\vec{r}\,' - \vec{v}(t_r) \frac{1}{c\mathscr{r}}\, \vec{\mathscr{r}} \cdot d\vec{r}\,' \Rightarrow d^3u = d\tau' \left(1 - \frac{\vec{v}(t_r) \cdot \vec{\mathscr{r}}}{c\mathscr{r}}\right)$$

$$\Phi(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \int d^3u \left(1 - \frac{\vec{v}(t_r) \cdot \vec{\mathscr{r}}}{c\mathscr{r}}\right)^{-1} \frac{1}{\mathscr{r}}\, \delta^3(\vec{u})$$

Peaks @ $\vec{u} = 0$, where $\vec{r}\,' = \vec{R}(t_r)$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{R}(t_r)|} \times \frac{1}{1 - \frac{\vec{v}(t_r) \cdot (\vec{r} - \vec{R}(t_r))}{c\, |\vec{r} - \vec{R}(t_r)|}}$$

$$\text{w/ } t_r = t - \frac{|\vec{r} - \vec{R}(t_r)|}{c}$$

- A similar calculation gives $\vec{A}$. So, for a moving point charge q w/ position $\vec{R}(t)$, the potentials are:

$$\Phi(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{R}(t_r)| - \frac{\vec{v}(t_r)}{c}\cdot(\vec{r}-\vec{R}(t_r))}$$

$$\vec{A}(\vec{r},t) = \frac{1}{c^2}\vec{v}(t_r)\,\Phi(\vec{r},t)$$

$$\text{w/ } t_r = t - \frac{|\vec{r}-\vec{R}(t_r)|}{c} \quad \& \quad \vec{v}(t_r) = \frac{\partial \vec{R}(t_r)}{\partial t_r}$$

- These seem relatively simple! However, it can be tricky to work out t_r. Let's look @ the simplest example, which is a charge q moving w/ constant velocity $\vec{v}$. We'll fix our coordinates & synchronize our clock so that it's @ the origin @ $t=0$.

$$\vec{R}(t) = \vec{v}\,t \quad \Rightarrow \quad t_r = t - \frac{1}{c}|\vec{r}-\vec{v}\,t_r| \quad \leftarrow |\vec{b}| = \sqrt{\vec{b}\cdot\vec{b}}$$

$$\hookrightarrow \quad c^2(t-t_r)^2 = (\vec{r}-\vec{v}\,t_r)\cdot(\vec{r}-\vec{v}\,t_r)$$

$$c^2t^2 - 2c^2 t\,t_r + c^2 t_r^2 = r^2 + v^2 t_r^2 - 2\vec{r}\cdot\vec{v}\,t_r$$

$$(c^2-v^2)\,t_r^2 + 2(\vec{r}\cdot\vec{v} - c^2 t)\,t_r + c^2t^2 - r^2 = 0$$

$$\hookrightarrow \quad t_r = \frac{\cancel{2}(c^2t-\vec{r}\cdot\vec{v}) \pm \sqrt{\cancel{4}(c^2t-\vec{r}\cdot\vec{v})^2 - \cancel{4}(c^2-v^2)(c^2t^2-r^2)}}{\cancel{2}(c^2-v^2)}$$

- The '−' sign in the quadratic formula gives $t_r < t$:

$$\hookrightarrow \quad t_r = \frac{t - \frac{1}{c^2}\vec{r}\cdot\vec{v} - \sqrt{\left(t-\frac{1}{c^2}\vec{r}\cdot\vec{v}\right)^2 + \left(1-\frac{v^2}{c^2}\right)\left(\frac{r^2}{c^2}-t^2\right)}}{1 - v^2/c^2}$$

- With some algebra we can use this to express Φ & $\vec{A}$ in terms of t, $\vec{r}$, & $\vec{v}$. For q w/ $\vec{R}(t) = \vec{v}t$:

$$\Phi(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)(r^2-c^2t^2) + \left(ct - \frac{1}{c}\vec{r}\cdot\vec{v}\right)^2}}$$

$$\vec{A}(r,t) = \frac{1}{c^2}\vec{v}\,\Phi(r,t)$$

Notice that these give $\Phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$, $\vec{A} = 0$ for $\vec{v} = 0$!

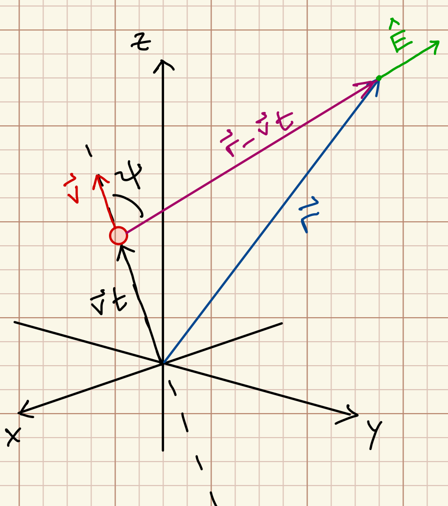
- You can see special relativity sneaking in here! Φ & $\vec{A}$ are really the 4 components of the "four-potential" A^μ ($\mu = 0,1,2,3$) w/ $A^0 = \Phi/c$, $A^1 = A_x$, $A^2 = A_y$, $A^3 = A_z$. Under a change of reference frame these components get mixed up just like the $t, x, y,$ & z components of x^μ.

- In the charge's frame of reference (which is an inertial frame, since I see it moving w/ $\vec{v}$ = constant) it isn't moving, so someone moving alongside it would say there's a Φ but no $\vec{A}$. (More on this later.)

- We can find $\vec{E}$ & $\vec{B}$ using $\vec{E} = -\vec{\nabla}\Phi - \partial\vec{A}/\partial t$ and $\vec{B} = \vec{\nabla}\times\vec{A}$, which gives:

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{\vec{r}-\vec{v}t}{|\vec{r}-\vec{v}t|^3} \frac{\left(1-v^2/c^2\right)}{\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}$$

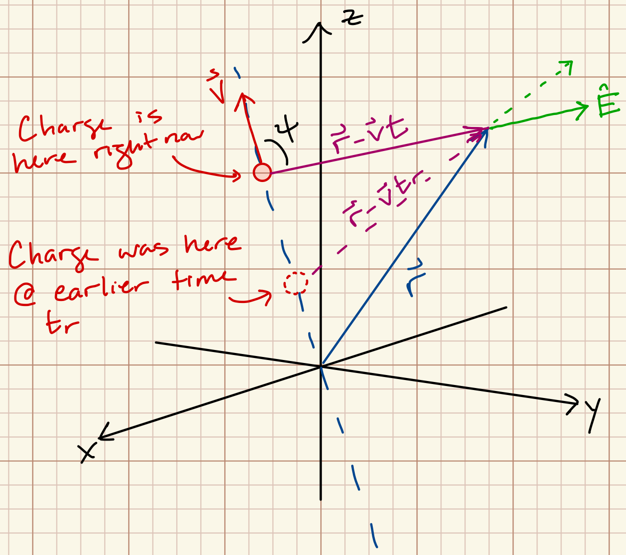
$$\vec{B}(\vec{r},t) = \frac{\mu_0 q}{4\pi} \frac{\vec{v}\times(\vec{r}-\vec{v}t)}{|\vec{r}-\vec{v}t|^3} \frac{\left(1-v^2/c^2\right)}{\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}$$



↑ $\vec{E}$ & $\vec{B}$ for a pt. charge w/ constant velocity (origin @ pos. of charge @ $t=0$)

- A few things to notice about $\vec{E}$ & $\vec{B}$.

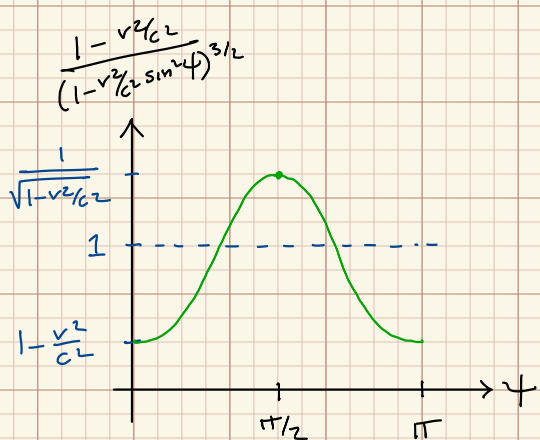
- First, even though potentials @ $\vec{r}, t$ depend on location of charge @ the earlier time t_r, the electric field is in the $\vec{r}-\vec{v}t$ direction and <u>not</u> the $\vec{r}-\vec{v}t_r$ direction.

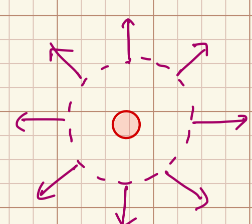


- Second, besides the "$\hat{r}/r^2$" part, $\vec{E}$ has a factor of

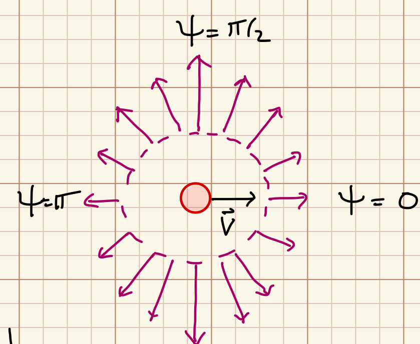
$$\frac{1 - v^2/c^2}{\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}$$

w/ $0 < 1 - v^2/c^2 < 1$ since $v < c$. Ahead of or behind q, $\vec{E}$ is smaller. At points perp. to $\vec{v}$ (ψ close to 90°) it is enhanced.





Charge @ rest, $|\vec{E}|$ depends only on distance from charge.



Moving charge, $|\vec{E}|$ depends on dist. from charge & direction b/t $\vec{r}$ & $\vec{v}$.

- Third, $\vec{B}$ is perp. to both $\vec{v}$ & $\vec{E}$. Because of the cross product $\vec{v}\times(\vec{r}-\vec{v}t)$, its magnitude can be written:

$$|\vec{B}(\vec{r},t)| = \frac{\mu_0}{4\pi}\frac{q\,v}{|\vec{r}-\vec{v}t|^2}\sin\psi\,\frac{(1-v^2/c^2)}{\left(1-\frac{v^2}{c^2}\sin^2\psi\right)^{3/2}}$$

Additional suppression @ $\vec{r}$ close to dir. of motion $\vec{v}t$.